Isometric uniqueness of a complementably universal Banach space for Schauder decompositions

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Abstract

We present an isometric version of the complementably universal Banach space $\mathbb{P}$ with a Schauder decomposition. The space $\mathbb{P}$ is isomorphic to Pełczyński's space with a universal basis as well as to Kadec' complementably universal space with the bounded approximation property.
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1 Introduction

A Banach space E is *complementably universal* for a given class of spaces if every space from the class is isomorphic to a complemented subspace of E. In 1969 Pełczyński [10] constructed a complementably universal Banach space with a Schauder basis. Two years later, Kadec [6] constructed a complementably universal Banach space for the class of spaces with the *bounded approximation property* (BAP). Just after, Pełczyński [9] showed that every Banach space with BAP is complemented in a space with a basis. Applying Pełczyński' decomposition argument [11], one immediately concludes that both spaces are isomorphic. One has to mention the result of Johnson & Szankowski [5] saying that no separable Banach space can be complementably universal for the class of all separable spaces.

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In this note we present a natural extension property that describes an isometric version of the Kadec-Pełczyński space. We present a construction of this space and show its isometric uniqueness. Most of the arguments are inspired by the recent works [8] and [7].

2 Preliminaries

Let X be a Banach space. A *Schauder decomposition*, called also a *finite-dimensional decomposition* (briefly: *FDD*) is a sequence $P_n \colon X \to X$ of finite rank pairwise orthogonal linear operators such that $x = \sum_{n=0}^{\infty} P_n x$ for every $x \in X$. Given such a decomposition, let $Q_n = P_0 + \dots + P_{n-1}$. Then Q_n is a finite-rank projection $Q_n \colon X \to X$. We shall say that X has *k-FDD*, whenever $k \geq \sup_{n \in \mathbb{N}} \|Q_n\|$. We shall actually consider 1-FDD only, which is usually called *monotone FDD* or *monotone Schauder decomposition*. Note that every Schauder decomposition is determined by finite-rank projections Q_n satisfying the conditions $Q_n Q_m = Q_{\min(n,m)}$ and $x = \lim_{n\to\infty} Q_n x$ for $x \in X$. We refer to [3, Chapter 4] for details.

In order to make some statements simpler, we shall consider linear operators of norm at most 1 only. In particular, given $\varepsilon > 0$, a linear operator $f \colon X \to Y$ is an *ε-isometry* if

$$(1-\varepsilon)\cdot\|x\| \leq \|f(x)\| \leq \|x\|$$

holds for every $x \in X$.

3 The crucial lemma

In this section we elaborate a lemma on "correcting" almost isometries which was the key fact in [8].

Lemma 3.1. *Let $f \colon X \to Y$ be an ε-isometry of Banach spaces. Then there exists a norm $\|\cdot\|_f$ on $X \oplus Y$ such that the canonical embeddings $i \colon X \to X \oplus Y$ and $j \colon Y \to X \oplus Y$ become isometric and furthermore the following conditions are satisfied:*

(1) $\|j \circ f - i\|_f \leq \varepsilon$.

(2) *Given a Banach space V, given linear operators $k \colon X \to V$, $\ell \colon Y \to V$ such that $\|\ell \circ f - k\| \leq \varepsilon$, there exists a unique linear operator $h \colon X \oplus Y \to V$ such that $h \circ i = k$ and $h \circ j = \ell$.*

Recall that all linear operators are non-expansive, therefore the key point in condition (2) is that $\|h\| \leq 1$.

Proof. Define

$$\|\langle x, y\rangle\|_f = \inf\Big\{\|x - w\|_X + \|y + f(w)\|_Y + \varepsilon\|w\|_X \colon w \in X\Big\},$$

where $\|\cdot\|_X$, $\|\cdot\|_Y$ are the norms of X and Y, respectively. It is easy to see that this is a norm on $X \oplus Y$ and the closed unit ball is the convex hull of the set

$$(B_X \times \{0\}) \cup (\{0\} \times B_Y) \cup G,$$

where B_X, B_Y are the closed unit balls of X and Y respectively, and

$$G = \{\langle w, -f(w)\rangle \colon w \in \varepsilon^{-1} B_X\}.$$

It is clear that the canonical embeddings i, j have norms ≤ 1. We need to check that they are indeed isometric.

Fix $x, w \in X$. We have

$$\|x - w\|_X + \|f(w)\|_Y + \varepsilon\|w\|_X \geq \|x - w\|_X + (1 - \varepsilon)\|w\|_X + \varepsilon\|w\|_X \geq \|x\|_X.$$

Passing to the infimum, we see that $\|\langle x, 0\rangle\|_f \geq \|x\|_X$. This shows that i is an isometric embedding. A similar argument shows that j is an isometric embedding.

By the definition of $\|\cdot\|_f$, condition (1) is satisfied. Concerning condition (2), given operators $k \colon X \to V$, $\ell \colon Y \to V$ such that $\|\ell \circ f - k\| \leq \varepsilon$, there is obviously a unique linear operator h satisfying $h \circ i = k$, $h \circ j = \ell$, just because of the linear structure. Note that the condition $\|\ell \circ f - k\| \leq \varepsilon$ implies that $h[G] \subseteq B_V$. Assuming $\|k\| \leq 1$ and $\|\ell\| \leq 1$, we see that $h[B_X \times \{0\}] \subseteq B_V$ and $h[\{0\} \times B_Y] \subseteq B_V$. We conclude that $\|h\| \leq 1$, because the unit ball of $\|\cdot\|_f$ is the convex hull of these three sets. $\square$

It is obvious that the statement above could be easily rephrased in the language of categories, saying that $X \oplus Y$ with the norm $\|\cdot\|_f$ and with the embeddings i, j is the initial object of a suitable category. The Banach space $\langle X \oplus Y, \|\cdot\|_f\rangle$ will be denoted briefly by $X \oplus_f Y$.

4 The category of projection-embedding pairs

We shall now prepare the setup for our construction. Note that a Banach space E with a monotone FDD can be described as the completion of an increasing chain $\{E_n\}_{n \in \omega}$ of finite-dimensional subspaces such that each E_n is 1-complemented in E_{n+1}. The projections $Q_n \colon E \to E_n$ are then defined inductively, as suitable compositions. This characterization will be used for constructing the universal space with a monotone FDD and showing its almost homogeneity.

We now define the relevant category $\mathfrak{K}$. The objects of $\mathfrak{K}$ are finite-dimensional Banach spaces. Given finite-dimensional spaces E, F, an $\mathfrak{K}$-arrow is a pair $\langle e, P\rangle$ of linear operators $e \colon E \to F$, $P \colon F \to E$, satisfying the following two conditions:

(P1) e is an isometric embedding.

(P2) $P \circ e = \mathrm{id}_E$, where E is the domain of e.

In particular, if $e\colon E \to F$ is the inclusion (i.e., $E \subseteq F$) then (P2) says that P is a projection of F onto E. In particular, E must be 1-complemented in F.

It is clear that such categories of pairs can be defined starting from an arbitrary category. Categories of pairs satisfying (P2) are known as *categories of projection-embedding pairs*, see [2] for further references.

The following amalgamation lemma can be found in [7]; its part involving embeddings belongs to the folklore (see e.g. [1]).

Lemma 4.1. *Let $\langle i, P\rangle\colon Z \to X$, $\langle j, Q\rangle\colon Z \to Y$ be $\mathfrak{K}$-arrows. Then there exist $\mathfrak{K}$-arrows $\langle i', P'\rangle\colon X \to V$, $\langle j', Q'\rangle\colon Y \to V$ such that $i' \circ i = j' \circ j$, $P \circ P' = Q \circ Q'$ and $j \circ P = Q' \circ i'$, $i \circ Q = P' \circ j'$ which is shown in the following diagram:*

$$\begin{tikzcd}
Y \arrow[r, shift left, "j'"] \arrow[d, shift left, two heads, "Q"] & V \arrow[l, shift left, two heads, "Q'"] \arrow[d, shift left, two heads, "P'"] \\
Z \arrow[u, shift left, "j"] \arrow[r, shift left, "i"] & X. \arrow[l, shift left, two heads, "P"] \arrow[u, shift left, "i'"]
\end{tikzcd}$$

The following approximation lemma will be used several times later.

Lemma 4.2. *Let $f\colon X \to Y$ and $T\colon Y \to X$ be linear operators of norm ≤ 1 such that*

$$(\ddagger) \qquad \|T \circ f - \mathrm{id}_X\| \leq \varepsilon.$$

Then f is an ε-isometry. Let $i\colon X \to X \oplus_f Y$, $j\colon Y \to X \oplus_f Y$ be the canonical isometric embeddings. Then there exist linear operators $P\colon X \oplus_f Y \to X$ and $Q\colon X \oplus_f Y \to Y$ such that

$$P \circ i = \mathrm{id}_X, \; Q \circ j = \mathrm{id}_Y, \; P \circ j = T, \; \textit{and} \; Q \circ i = f.$$

Proof. Given $x \in X$, we have

$$\varepsilon\|x\| \geq \|Tf(x) - x\| \geq \|x\| - \|Tf(x)\| \geq \|x\| - \|f(x)\|,$$

which shows that $\|f(x)\| \geq (1-\varepsilon)\|x\|$, that is, f is an ε-isometry.

Now, applying Lemma 3.1 with $k := \mathrm{id}_X$, $\ell := T$, we obtain a linear operator $P\colon X \oplus_f Y \to X$ satisfying $P \circ i = \mathrm{id}_X$ and $P \circ j = T$. Applying Lemma 3.1 again with $k := f$, $\ell := \mathrm{id}_Y$, we obtain a linear operator $Q\colon X \oplus_f Y \to Y$ satisfying $Q \circ i = f$ and $Q \circ j = \mathrm{id}_Y$. $\square$

The result above says that, given an $\mathfrak{L}$-arrow $\langle f, T\rangle$ satisfying ($\ddagger$), there exist $\mathfrak{K}$-arrows $\langle i, P\rangle$, $\langle j, Q\rangle$ such that $\langle i, P\rangle$ is ε-close to $\langle j, Q\rangle \circ \langle f, T\rangle$.

Recall that a Banach space E is *rational* if $E = \mathbb{R}^n$ with a norm such that its unit ball is a polyhedron spanned by finitely many vectors whose all coordinates are rational numbers. An operator $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is *rational* if $T[\mathbb{Q}^n] \subseteq \mathbb{Q}^m$. Finally, let us call a $\mathfrak{K}$-arrow *rational* if both of its components are rational operators.

The following two facts are easily proved by standard "approximation" arguments. We omit the details.

Lemma 4.3. *Let E be a finite-dimensional Banach space. Then for every $\varepsilon > 0$ there exists an $\mathfrak{L}$-arrow $e\colon P \to V$ into a rational Banach space V, such that*

$$\|P \circ e - \mathrm{id}_E\| \leq \varepsilon.$$

Lemma 4.4. *Let V be a rational Banach space, $\varepsilon > 0$, and let $\langle e, P\rangle\colon V \to E$ be a $\mathfrak{K}$-arrow. Then there exist an $\mathfrak{L}$-arrow $\langle f, T\rangle\colon E \to W$ and a $\mathfrak{K}$-arrow $\langle i, Q\rangle\colon V \to W$ such that W is a rational Banach space, i, Q are rational operators, and the following inequalities hold:*

$$\|f \circ e - i\| \leq \varepsilon \qquad \text{and} \qquad \|P \circ T - Q\| \leq \varepsilon.$$

5 The construction

Denote by $\mathcal{F}$ the subcategory of $\mathfrak{K}$ consisting of all rational $\mathfrak{K}$-arrows. Obviously, $\mathcal{F}$ is countable. Looking at the proof of Lemma 4.1, we can see that $\mathcal{F}$ has the amalgamation property. We now use the concepts from [7] for constructing a "generic" sequence in $\mathcal{F}$. First of all, a *sequence* in a fixed category $\mathfrak{C}$ is formally a covariant functor from the set of natural numbers ω into $\mathfrak{C}$. Up to isomorphism, every sequence in $\mathfrak{K}$ corresponds to a chain $\{E_n\}_{n\in\omega}$ of finite-dimensional subspaces, together with projections $P^n_m\colon E_n \to E_m$ satisfying $P^n_m \circ P^m_k = P^n_k$ for every $k < m < n$. The sequence $\{P^n_m\}_{m<n<\omega}$ induces a sequence of projections $P_n\colon E \to E_n$, where E is the completion (formally, the co-limit) of $\bigcup_{n\in\omega} E_n$, satisfying $P_n \circ P_m = P_{\min(m,n)}$ for every $m, n \in \omega$. By this way, the 1-FDD of a Banach space X is translated into the existence of a sequence in $\mathfrak{K}$ whose co-limit is X. For the sake of convenience, we shall denote a sequence by $\vec{X}$ (or $\vec{Y}$, $\vec{U}$, etc.), having in mind a chain $\{X_n\}_{n\in\omega}$ of finite-dimensional spaces together with the appropriate projections between them. Given $X_n \subseteq X_m$, the $\mathfrak{K}$-arrow $\langle i, P^m_n\rangle$, where i is the inclusion, will be called the *bonding arrow* from X_n to X_m. When looking at $\vec{X}$ as a functor from ω, the bonding arrows are just images of the pairs $\langle m, n\rangle$ (with $m \leq n$).

Following [7], we shall say that a sequence $\vec{U}$ in $\mathcal{F}$ is *Fraïssé* if it satisfies the following condition:

(A) Given $n \in \omega$, and an $\mathcal{F}$-arrow $f\colon U_n \to Y$, there exist $m > n$ and an $\mathcal{F}$-arrow $g\colon Y \to U_m$ such that $g \circ f$ is the bonding arrow from U_n to U_m.

It is clear that this definition is purely category-theoretic. The name "Fraïssé sequence", as in [7], is motivated by the model-theoretic theory of Fraïssé limits explored by Roland Fraïssé [4]. One of the results in [7] is that every countable category with amalgamations has a Fraïssé sequence. For completeness, we present the argument in our special case.

Theorem 5.1 ([7])**.** *The category $\mathcal{F}$ has a Fraïssé sequence.*

Proof. Throughout the proof, we assume only that $\mathcal{F}$ is a countable category with the amalgamation property and with the initial object 0 (which is the trivial space). Let $\Delta = \{\langle m, n\rangle \in \omega \times \omega \colon m \leq n\}$. Given a sequence $\vec{x} \colon \omega \to \mathcal{F}$, we denote by x_n the nth object and by x_m^n the bonding arrow from x_m to x_n. Now, formally speaking, a sequence (i.e., a covariant functor) can be regarded as a suitable function from Δ into $\mathcal{F}$, since the objects are uniquely determined by the bonding arrows. Note that the set $S \subseteq \mathcal{F}^\Delta$ of all covariant functors is closed in the space $\mathcal{F}^\Delta$ endowed with the product topology. Given $n \in \omega$ and an $\mathcal{F}$-arrow $f \colon a \to b$, define

$$V_{f,n} = \{\vec{x} \in S \colon x_n = a \implies (\exists\, m > n)(\exists\, g)\ \ g \circ f = x_n^m\}.$$

It is clear that $V_{f,n}$ is an open set. We show that it is dense.

A basic nonempty open set $U \subseteq S$ is determined by a finite sequence of $\mathcal{F}$-arrows $\vec{s} = \{s_i^j\}_{i \leq j < k}$ such that each s_i^i is an identity and $s_j^\ell \circ s_i^j = s_i^\ell$ whenever $i < j < \ell < k$. The open set $U = U(\vec{s})$ consists of all sequences that extend $\vec{s}$. Fix f and n as above. Enlarging $\vec{s}$ if necessary and at the same time shrinking U, we may assume that $k > n$. Now, if $a \neq s_n$, then $U \subseteq V_{f,n}$. Suppose $a = s_n$. Using the amalgamation property, we find $\mathcal{F}$-arrows $h \colon s_{k-1} \to w$ and $g \colon b \to w$ such that $h \circ s_n^{k-1} = g \circ f$. Let $\vec{t}$ be the sequence $\vec{s}$ extended by h (so its length is $k+1$ and $t_k = w$). Now $U(\vec{t}) \subseteq U \cap V_{f,n}$, showing that $U \cap V_{f,n}$ is nonempty. This shows that $V_{f,n}$ is dense in S. By the Baire Category Theorem, there is $\vec{u} \in \bigcap_{f \in \mathcal{F}, n \in \omega} V_{f,n}$. Just by the definition of $V_{f,n}$, it follows that $\vec{u}$ is a Fraïssé sequence. $\square$

From now on, we fix a Fraïssé sequence $\{U_n\}_{n\in\omega}$ in $\mathcal{F}$. As usual, we assume that the embeddings are inclusions. We shall denote by P_n^m the bonding projection from U_m onto U_n. Let $\mathbb{P}$ be the completion of the union $\bigcup_{n\in\omega} U_n$. We shall denote by P_n the projection from $\mathbb{P}$ onto U_n induced by the sequence $\{P_n^m\}_{n \leq m < \omega}$.

We shall prove, in particular, that $\mathbb{P}$ is isomorphic to the Kadec-Pełczyński complementably universal space for Schauder bases. The next sections are devoted to proving its isometric properties, like universality and uniqueness.

6 Universality

Theorem 6.1. *Let X be a Banach space with a monotone FDD. Then there exists an isometric embedding $e \colon X \to \mathbb{P}$ such that $e[X]$ is 1-complemented in $\mathbb{P}$.*

Proof. Fix a Banach space X with 1-FDD and let this be witnessed by a chain $\{X_n\}_{n\in\omega}$ together with suitable projections $Q_n \colon X \to X_n$. We let $Q_n^m = Q_n \restriction X_m$ for $m > n$. We construct inductively linear operators $e_n \colon X_n \to U_{k_n}$, $R_n \colon U_{k_n} \to X_n$ such that

(1) $\|R_n \circ e_n - \mathrm{id}_{X_n}\| < 2^{-n}$,

(2) $\|e_{n+1} \restriction X_n - e_n\| < 2^{-n}$,

(3) $\|R_{n+1} \restriction U_{k_n} - R_n\| < 2^{-n}$.

Recall that, according to our previous agreement, all linear operators have norm ≤ 1. We may assume that $X_0 = U_0 = \{0\}$, therefore it is clear how to start the induction. Suppose e_n and R_n (and $k_n \in \omega$) have already been defined.

Note that $\langle e_n, R_n \rangle$ is an $\mathfrak{L}$-arrow. By Lemma 4.2, there exist $\mathfrak{K}$-arrows $\langle i, S \rangle \colon X_n \to W$ and $\langle j, T \rangle \colon U_{k_n} \to W$, where $W = X_n \oplus_{e_n} U_{k_n}$, and the following conditions are satisfied:

(4) $T \circ i = e_n$ and $S \circ j = R_n$,

(5) $\|j \circ e_n - i\| < 2^{-n}$,

Using Lemma 4.1, we may further extend W so that there exists also a $\mathfrak{K}$-arrow $\langle \ell, G \rangle \colon X_{n+1} \to W$ satisfying

(7) $\ell \restriction X_n = i$,

(8) $Q_n^{n+1} \circ G = S$.

Recall that U_n is a rational Banach space. Thus, using Lemma 4.4, we can extend W further, so that the extended arrow from U_n to W will become rational. Doing this, we make some "error" of course, although we can still preserve (6), (7) and we can also preserve (4)–(5), because all the inequalities appearing there are strict. Now we use the fact that $\{U_n\}_{n \in \omega}$ is a Fraïssé sequence. Specifically, we find $k_{n+1} > k_n$ and rational operators $g \colon W \to U_k$ and $H \colon U_{k_{n+1}} \to W$ such that $H \circ g = \mathrm{id}_W$, $T \circ H = P_{k_n}^{k_{n+1}}$, and $g \circ j$ is the inclusion $U_{k_n} \subseteq U_k$.

Define $e_{n+1} = g \circ \ell$ and $R_{n+1} = G \circ H$. Using (4)–(7), it is straightforward to check that conditions (1)–(3) are satisfied. This finishes the inductive construction.

Passing to the limits, we obtain linear operators $e \colon X \to \mathbb{P}$ and $R \colon \mathbb{P} \to X$. Conditions (1)–(3) imply that e is an isometric embedding and $R \circ e = \mathrm{id}_X$. In particular, $e[X]$ is 1-complemented in $\mathbb{P}$. $\square$

Corollary 6.2. *The space $\mathbb{P}$ is isomorphic to Pełczyńki's complementably universal space for Schauder bases, as well as to Kadec's complementably universal space for the bounded approximation property.*

Proof. By the result of Pełczyński [9], the complementably universal space constructed by Kadec [6] is isomorphic to the complementably universal space constructed by Pełczyński. On the other hand, the well-known Pełczyński decomposition method [11] implies that there is, up to isomorphism, only one complementably universal Banach space for monotone FDD (as well as for other related classes). Thus, in view of Theorem 6.1, our space $\mathbb{P}$ is isomorphic to Pełczyński's space. $\square$

7 Isometric uniqueness and homogeneity

In this section we show further properties of the space $\mathbb{P}$. In order to shorten some statements, let us say that a space Y is *ε-complemented* in X if $Y \subseteq X$ and there is a linear operator $T\colon X \to Y$ satisfying $\|Ty - y\| \leq \varepsilon\|y\|$ for every $y \in Y$. In particular, "0-complemented" means "complemented", i.e., there is a projection $P\colon X \to Y$ (recall again that all operators have norms ≤ 1). We shall say that f is a *$(< \varepsilon)$-embedding* if it is an ε'-isometric embedding for some $0 < \varepsilon' < \varepsilon$. Similarly, we shall say that Y is *$(< \varepsilon)$-complemented* in X if it is ε'-complemented for some $0 < \varepsilon' < \varepsilon$.

Let us consider the following extension property of a Banach space X:

(E) Given a pair $E \subseteq F$ of finite-dimensional Banach spaces such that E is complemented in F, given an isometric embedding $i\colon E \to X$ such that $i[E]$ is complemented in X, for every $\varepsilon > 0$ there exists an ε-isometric embedding $g\colon F \to X$ such that $\|g \restriction E - i\| < \varepsilon$ and $g[F]$ is ε-complemented in X.

Lemma 7.1. *Assume X has a monotone FDD and satisfies condition (E). Then, given $\varepsilon, \delta > 0$, given finite-dimensional spaces $E \subseteq F$, given a $(< \varepsilon)$-isometric embedding $f\colon E \to X$ such that $f[E]$ is $(< \varepsilon)$-complemented in X, there exists a δ-isometric embedding $g\colon F \to X$ such that $\|g \restriction E - f\| < \varepsilon$ and $g[F]$ is δ-complemented in X.*

Proof. Correcting f if necessary, we may assume that $f[E] \subseteq A$ for some finite-dimensional complemented subspace A of X. Here we have used the fact that X has 1-FDD, that is, X has a chain of complemented finite-dimensional subspaces whose union is dense. Using Lemma 4.2, we can find isometric embeddings $i\colon E \to V$, $j\colon A \to V$, where V is a finite-dimensional space, $i[E]$ and $j[A]$ are complemented in V and $\|j \circ f - i\| \leq \eta$, where $\eta < \varepsilon$ is such that f is an η-embedding and $f[E]$ is η-complemented in A. Using Lemma 4.1, we may further extend V so that there is also an isometric embedding $k\colon F \to V$ with the property that $k[F]$ is complemented in V. Applying property (E) to the inverse of j, we find a δ-isometry $e\colon V \to X$ such that $e[V]$ is δ-complemented in X and $\|e \circ j - \mathrm{id}_A\| < \varepsilon - \eta$. Finally, $g = e \circ k$ is the required δ-embedding. $\square$

Theorem 7.2. *Let $\mathbb{P}$ and $\mathbb{K}$ be Banach spaces satisfying condition (E) and let $h\colon A \to B$ be a bijective linear isometry between complemented finite-dimensional subspaces of $\mathbb{P}$ and $\mathbb{K}$, respectively. Then for every $\varepsilon > 0$ there exists a bijective linear isometry $H\colon \mathbb{P} \to \mathbb{K}$ that is ε-close to h. In particular, $\mathbb{P}$ and $\mathbb{K}$ are linearly isometric.*

Proof. We first "move" h so that, for some $\varepsilon_0 < \varepsilon$, its domain becomes a $(< \varepsilon_0)$-complemented subspace of some complemented finite-dimensional space $A_0 \subseteq \mathbb{P}$ and its range becomes a $(< \varepsilon_0)$-complemented subspace of some complemented

finite-dimensional space $B_0 \subseteq \mathbb{K}$. Moreover, after "moving", h becomes a $(< \varepsilon_0)$-embedding. Let $\{\varepsilon_n\}_{n\in\omega}$ be strictly decreasing and such that

$$2\sum_{n=1}^{\infty} \varepsilon_n < \varepsilon - \varepsilon_0.$$

We shall construct inductively linear operators $f_n \colon A_n \to B_n$, $g_n \colon B_n \to A_{n+1}$ such that the following conditions are satisfied:

(0) f_0 extends h.

(1) A_n is a finite-dimensional complemented subspace of $\mathbb{P}$.

(2) B_n is a finite-dimensional complemented subspace of $\mathbb{K}$.

(3) $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$.

(4) f_n and g_n are $(< \varepsilon_n)$-embeddings whose images are $(< \varepsilon_n)$-complemented in $\mathbb{K}$ and $\mathbb{P}$, respectively.

(5) $\|g_n \circ f_n - \mathrm{id}_{A_n}\| < \varepsilon_n$.

(6) $\|f_{n+1} \circ g_n - \mathrm{id}_{B_n}\| < \varepsilon_n$.

(7) $\bigcup_{n\in\omega} A_n$ is dense in $\mathbb{P}$ and $\bigcup_{n\in\omega} B_n$ is dense in $\mathbb{K}$.

In each of the spaces $\mathbb{P}$ and $\mathbb{K}$ we fix an increasing sequence of complemented finite-dimensional subspaces, with dense unions. The spaces A_n and B_n will be taken from those sequences. By this way we ensure condition (7).

Once we have defined f_n and g_n, we use Lemma 4.2 and property (E) twice, in order to obtain f_{n+1} and g_{n+1}. Thus, the construction can be carried out.

Finally, the sequence $\{f_n\}_{n\in\omega}$ converges to a linear operator whose completion is an isometric embedding $f_\infty \colon \mathbb{P} \to \mathbb{K}$. Similarly, $\{g_n\}_{n\in\omega}$ converges (after taking the completion) to an isometric embedding $g_\infty \colon \mathbb{K} \to \mathbb{P}$. Furthermore, $f_\infty \circ g_\infty = \mathrm{id}_{\mathbb{K}}$ and $g_\infty \circ f_\infty = \mathrm{id}_{\mathbb{P}}$, showing that these operators are bijective isometries. Setting $H = f_\infty$ and applying condition (0), we obtain the desired isometry. □

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